Theoretical Note
FOR OURSELVES
December 17, 1973
Feanör, 2.

STATE-SPACE

"State-Space" is a theoretical tool used in the "interdisciplinary field" known as "Systems Theory", which emerged out of the ferment in bourgeois scientific ideologies in this century. I believe we can ~~pillage~~ it for our own purposes of dialectical theorizing and exposition in the creation of a powerful "mathematics of dialectics". By this phrase I mean a highly condensed notational language, with a high degree of meaning per symbol, which we can use to make visible complex interactions and to display "univocally" the intricate interconcatenation of processes in a dialectical "concrete totality" or "system", [even] ~~i.e.~~ in a single formula. (It is "mathematical" in the sense that it is a language of quantity, a language that gets a handle on processes through their quantitative aspects, but which also includes or at least reaches out toward their qualitative aspects, which always reveals at very least the quantitative shadow of the qualitative. We will see how this can be.)

I want to show simply, clearly, and step by step what the state-space technique is and how to use it in constructing "mathematical models" of totalities. What I will be saying, in sum, is the following: Rational mechanics is the formal science of motion in space; mechanical motion. Evolution, development -- change in general -- is movement, motion, but not mechanical motion; not simply motion in space. In order to develop a [formal] science of evolution from the basis already conquered in physics, in mechanics -- i.e. a mechanics of evolution in general and a mechanics of the organic or of organism -- it is necessary to develop a space of states. Evolution is change of state. Mechanical motion is change-of-place in physical space. Evolution is change-of-place in state-space.

The state-space concept is just a formalization, a rigourous making-sense out-of, certain everyday notions visible in common usages and speech practices. When we use expressions such as "He's going places"; "Where are you going in life?"; "It's a stinking shame where this society is going"; "Where is humanity going in the twentieth century?", etc. we are using an idiom which the state-space concept makes explicit sense out of. For example, when Trotsky titles a pamphlet "Where is Britain going" or "Whither France?", he's obviously not talking about spatial motion, as if part of the European continent were going to pick up and move, say, to the top of the world; the arctic circle -- not even under the deGualle regime! Yet he's obviously talking about movement, and about change that has a direction. It is not motion in space, but motion in state that he is talking about; motion which has a direction in state terms.

We can start to develop the concept of "state-space" by first treating the concepts of "state" and "space" separately.

Now, what precisely do we mean by the concept of the "state" of a system? State-space theory gives a quantitative answer (though, as we shall see, this quantitative answer is not without its qualitative implications). We will defer until later the discussion as to whether this can ever be a sufficient answer. For state-space purposes, at any rate, the state of a system is represented as a ordered tuple set of quantities, these quantities being the essential measures of the system in question; those measurements which measure the essence of the system in question; which 'characterize' it -- which give its 'characteristic'. Each of these measures is called a state-variable of the system.

Now obviously, this state-set could be an infinite set; could contain an indefinite number of measures as elements. But usually, for limited purposes, a special, finite subset will do to characterize the system, to give its 'fingerprint' or 'signature' -- will give a more than adequate 'reading' or evaluation of the system. For instance, when you want to know whether or not you have an infection, the only state-variable regarding the state of your body-system you may care about, that may be relevant, is your temperature. The state-set for your body for you at that time Call it (t_o) is (T) or $(\,T(t_o)\,)$ * where T = temperature in degrees Fahrenheit -- the state-set consists of a single element, the quantity T. Say translate "Temp. at time t_o" $T(t_o) = 99.8^{\circ}$. Then the state of your body-system at time t_o is (99.8), according to this characterization.

In sum; what are the essential measures of a system has to be determined, in the first place, egoistically, i.e., in accordance with the intensionality of the subject designing the state-space, and the mathematical model -- a determination which no doubt may differ to a certain extent with different subjects and different projects. But neither, in the second place, is this intensionality merely arbitrary with regard either to the subject or the object (system) of this practice of modeling, nor is the genesis of the guiding intensionality somehow "outside" its material.

Having addressed the concept of "state", we now turn to the concept of "space". What we need in order to make our modeling possible, is a "Space" in which our system's various states can "take place" as it were; the space in which each and every possible state of a given system can occurr, can "find room' for itself, for as "long" or as "far" as we care to follow its development; its existence -- perhaps for its entire duration. We want a

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*(read "T of t_o" or "T at time t_o").

space composed of points, in which every "point" is associated with a specific, unique "state" of the system in question.

That's easy for the example with the thermometer. All we need is a single "number-line" -- a numbered line of points (that is, an infinite stack of points, with each point associated with a real number, and in this case, with the whole number-associated points numbered explicitly, at a regular interval), like this:

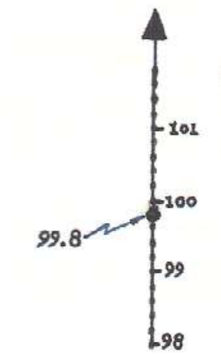


Figure 1.

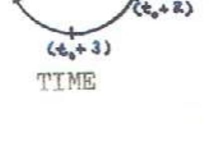
In fact, it looks a lot like a real thermometer. A thermometer is a kind of palpable state-space, with which we can "read" our state-of-health at a time t_o, in terms of fever and infection, simply by using it appropriately at time t_o. The level of the mercury in the glass tube then forms a -- vastly abbreviated -- of physical analog? scale? mathematical model of our health at that time. This single number-line, as an aggregation of points, is, for the mathematical concept of "space", to be classified as a space, and specifically as a one-dimensional space. This classification may appear to violate our every-day conception, for we tend to think of space only in terms of the three-dimensional space we are most used to. But in mathematics the concept of space is generalized to include spaces of any number of dimensions, 0, 1, 2, 3, 4 and so on all the way up to infinite-dimensional space, which is named "Hilbert space". (cf. Dr. Muses negadimensions and fractional dimensions (the concept of a DIMENSIONAL CONTINUUM))

One other thing to notice about Figure 1. : here the state of the body system is represented solely as a point. There is no information here as to the direction of the system in terms of health, or as to whether the fever is rising or falling. The state-direction is not indicated. A single value of a state-variable, such as 99.8° in this example, which conveys no directional information, is called a scalar. Think of the state-space pictured above in Figure 1. as a number scale and you will grok where the name comes from. The scalar -- the simple number, [the bare quantity] -- is about the simplest of the "mathematical objects" -- that is, the mental toys -- that mathematicians play around with. It is the most abstract, barren; least informative, least concrete of these internal toys. It is the basic object used in arithmetic. [But mathematics ~~does not~~ but does begin end with arithmetic. cf. Dr. Muses; G.S. Brown]

Suppose we want something richer? Suppose we want a mathematical object to play around with that conveys both numeric and directional information simultaneously? Sure we can have it! The mathematicians, in fact, have already thought of it; they call it the vector. They define it as a "quantity" which has both magnitude (quantity) and direction. Ah! One more step toward the concrete! This mental object even begins to take on some of the shape of and to remind one of some of the objects, the toys outside -- to "model" them, in fact. But shooosh! You're not supposed to notice that -- keep that kind of dirty-Jewish thinking to yourself!

So, our "quantities" begin to take on shades of qualitative information. Directionality, no less! We can represent our state, (99.8), as a vector, once we have determined the direction, in terms of temperature, in which our body-system is moving. It can go, in this one-dimensional space, in only two possible directions; "up" or "down", that is, toward increasing temperature or toward decreasing temperature. To find out this direction, we have either to feel ourselves; to feel whether our temperature is rising or falling, or to take a state-reading again, at a slightly later time, say $t_o + 1$ minute. If $T(t_o + 1) = 100^o$, then our direction-of-state is +T (rising), whereas, if $T(t_o + 1) = 99^o$ say, then our system-direction is -T (falling), as depicted in Figure 2a. and Figure 2b. respectively, below. Whereas we picture a scalar as a point, we picture the vector as an arrow pointing in the appropriate direction, and with a length corresponding to the magnitude of the vector. (pardon me if I lay our space on its side for a moment, so as to "save space" of another sort! Fear not! This changes nothing. Our space is "invariant" (translate: "in-different") to this rotation):

omits oscillatory (periodic) case



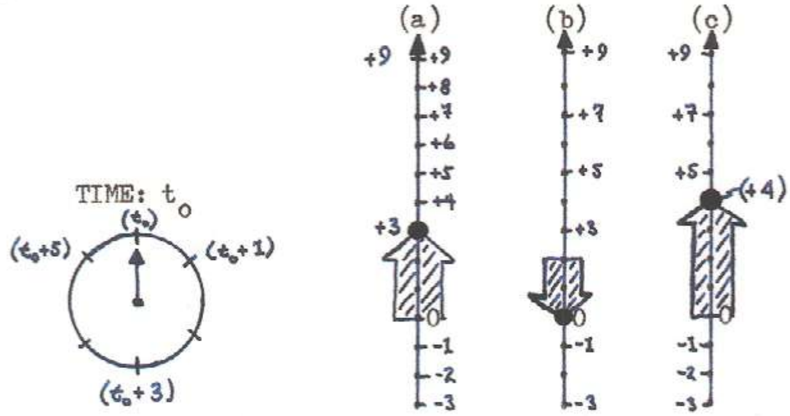
TIME

Figure 2a. if $T(t_o+ 1) = 100$ Figure 2b. if $T(t_o+1) = 99$

The length of the vectors in either case is 99.8^o, because that is the magnitude of the vector at time t_o, but the arrows point in opposite directions, because in case (a) the temperature is headed up the scale, whereas in case (b) it is headed down.

Vectors occurr in all sorts of spaces. A vector that occurrs in a state-space is called a state-vector.

Well, this one-dimensional state-space is all fine and dandy for simple problems, simple systems, or simple intensionalities, concerned with only one measurement ~~of~~ or "determination"* of a system. But suppose we have a more complicated system, or a more complicated interest in a system, so that a single state-variable like T is no longer sufficient to "determine" the system for us; to tell us what we desire to know? Suppose we have a system for which we pay attention to 2 or even 3 different state-variables, call them $a(t)$, $b(t)$, and $c(t)$. How do we represent <u>this</u> system? Well, we have here not one but <u>three</u> number-lines, and we want to represent the state of the system at a specific time $t = t_o$ say, a state which we have defined to be the set ($a(t_o)$, $b(t_o)$, $c(t_o)$), and we want to represent it <u>unitarily</u>; to be able to "see" the state as a single picture, <u>all at once</u>. Suppose $a(t_o) = 3$; $b(t_o) = 2$; and $c(t_o) = 4$. Then we can put the three number lines together into one picture thusly:



$S(t_o) = (3,2,4)$.

<u>Figure 3.</u>

So, our state at time $t_o = S(t_o) = (a(t_o), b(t_o), c(t_o)) = (3,2,4)$, with 3, 2, and 4, being measured in whatever units are appropriate to a, b, and c respectively, be they °F., angstroms, ergs, feet, liters, grams, apples, oranges, or what have you. As depicted above, at time t_o vector <u>a</u> is rising, vector <u>b</u> is falling, and vector <u>c</u> is rising. Thus we immediately come to one of the problems of this way of combining the three number lines: our depiction loses all unity of direction. Represented this way, our system has no one, singular direction, but many, perhaps contradictory directions, one direction per state-variable.

What we have here is not one state-space, with one point per state (i.e. with each point representing one and only one state of the system), but <u>three separate</u> state-spaces side-by-side, i.e., jumbled together. This representation is complicated, sloppy, repetitious, and difficult to read at a glance -- it makes it hard to "see" the state of the system. We have to worry about 3 different

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*(in Hegelian terms, the state-variables, or 'variables-of-state' correspond to the <u>determinations</u> of the system, quantitatively expressed, or "measured").

points in order to get a take on a single state of the system. Whereas, what we asked for originally was a "one-to-one correspondence" between points and states. Are we going to settle for a "three-to-one" correspondence? [Fuck that shit!] Let's go on. How can we make those 3 points into one?

Well, [fuck,] isn't there a better way of putting these three number-lines together? Bunching them up close together into a sheath will only make things worse. We have three spaces, and we want to make them one space. We've got to connect them somehow together. To be connected, they have to have at least part of their space in common; they've got to share points. But if we have them sharing points, how do we keep from blurring them together. If we don't preserve their individual identity in this combining, they're useless to us. Well, let's experiment; let's start with the minimum: the least connection they can have is <u>one point in common</u>. Now, all their other points, the ones numbered 4, 9, 120, 18,000, etc., are measured in different units, and mean different things. But there's one number that means the same, no matter what units its measured in -- <u>zero</u>. So let's join them only at their zero-points. Also, the three number lines could share this point and still assume any one of an infinite number of "attitudes" toward one another, going off every which-way at crazy angles. [For (linear) independence's sake
For simplicity sake, let's keep things "square" -- all three lines mutually perpendicular, that is, at <u>right</u> angles to eachother;]

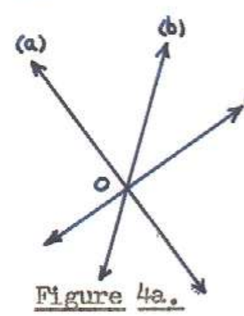


Figure 4a.

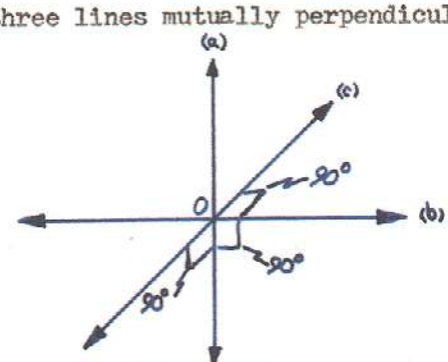


Figure 4b.

Figures 5. show how to locate the point in this space corresponding to our state (3,2,4) at time t_0:

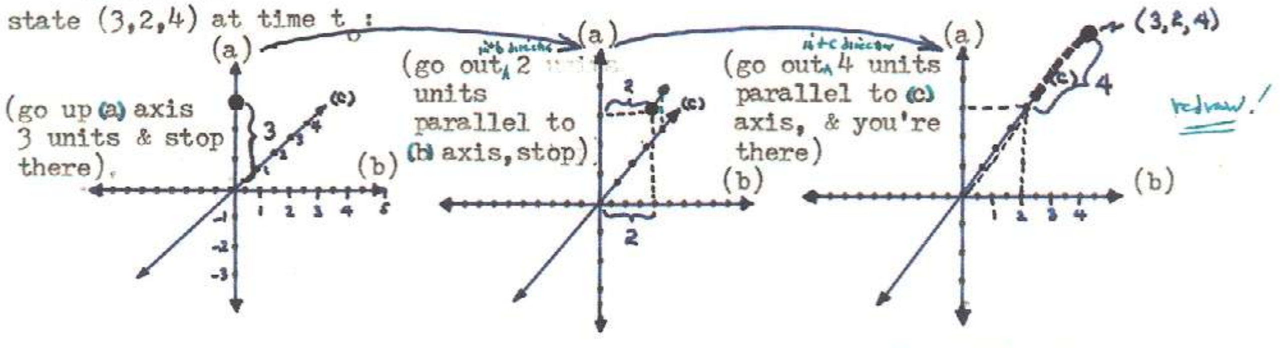
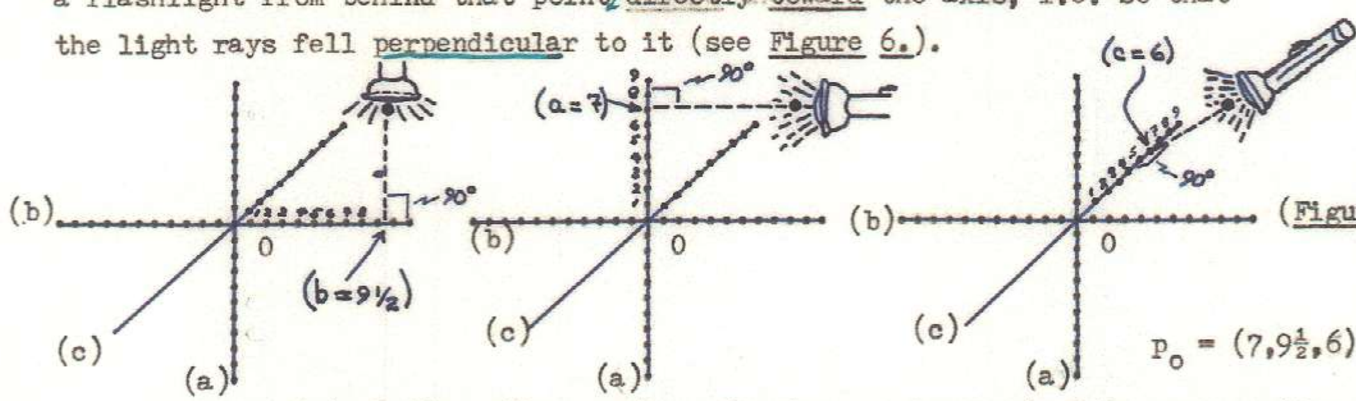


Figure 5a. Figure 5b. Figure 5c.

So, to sum up: all three of these "spaces", a, b, and c, now share a single point in common, namely their zero-point, called the "origin". The three one-dimensional state-spaces have become united as one three-dimensional state-space. Each state-line is perpendicular to the other two, all of them together thus forming three "orthogonal" or "normal" axes of what is known as a [rectangular Cartesian (after Descartes)] coordinate system. "Coordinate" refers to the fact that any point in the space -- call it p_o -- can be located uniquely by specifying a set of three numbers, one for each axis, each of which is known as a "coordinate" of that point. For the case of point p_o, we might represent that set as (a_o, b_o, c_o). The coordinate for each axis, for a given point, is the number associated with the projection of the point in question onto that axis, i.e. the place on the axis where the shadow of the point would fall if you shined a flashlight from behind that point ~~directly~~ perpendicularly toward the axis, i.e. so that the light rays fell perpendicular to it (see Figure 6.).



(Figures 6.)

At this point, let's notice another advantage we have gained by representing our system in this way. We are no longer confined to showing only one state "at a time". We don't have to redraw our whole state-picture for each moment of time, as before, in Figures 1, 2, and 3,. We can measure, at regular time intervals, the states of our system in terms of its three state-variables, a(t), b(t), and c(t) -- by direct observation, with instruments, by looking up in a published table of data, or by calculation, depending on the nature of the system and of our knowledge about it. Each observed state is thus converted into the three coordinates of a single point in our state-space. We can put in and line up each of these points in the order of their succession in time. or rather, of the succession in time of the system-states which they represent. We then draw a single line or "curve" connecting all these points together in the order of their succession. We assume that every point on this curve, not just the points originally filled in from observation, represent states of the system -- that the curve is "everywhere-dense" with state-points. This simply means that we assume that all the points on the curve in-between any two measured, observed points -- an infinite number -- represent states which the system "passed-through"

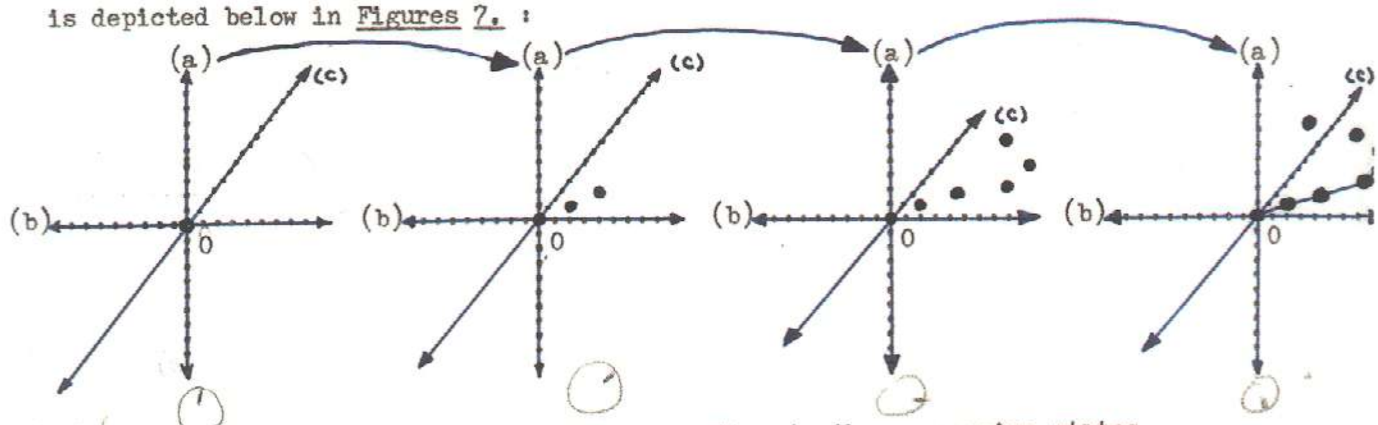
also can [illegible] [illegible] and/or oscillations or other movements

we assume that the in-between points are the states at the in-between times.

define/expand

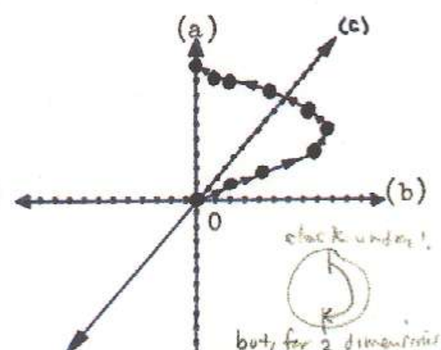
in-between our observations; "instantaneous" states which occurred at instants when we were not measuring. This also means we assume that no sudden deviations of direction and magnitude of the state-variables have taken place between observations, i.e. that the measured points should be connected like this and not like this . Such an assumption is in many cases unwarranted, in which cases the lines so drawn would represent only an approximation of the actual "walk" of the system through its state space. Only more detailed observation of the states of the system, at more frequent intervals, or a fuller understanding of its laws, translated into continuous equations which can be used to calculate the values of its state-variables for any instant of time, can remedy this situation.

The resulting locus of state-points or "curve" is a representation of all the "changes" or "appearances" which the system goes through during the observation period or ~~and~~ throughout its existence. This curve has a name; it is called the "state-space trajectory" of the system. The state-space itself is the locus of all the potential, conceivable values of the state-variables. The state-space trajectory is the locus of all the actualized state-points of the system as a whole, [as a totality of its state-variables] -- a representation of all the states which the system actually "assumes" (becomes or (momentarily) is) in the totality of its "moments". This whole process is depicted below in Figures 7. :



STEP 1 : fill-in the points corresponding to the successive states.

(t)	a(t)	b(t)	c(t)
0	0	0	0
1	1	2	$\frac{1}{2}$
2	2	4	1
3	3	8	2
4	4	9	6
5	5	8	8



(cont'd):

(t)	a(t)	b(t)	c(t)
6	6	5	6
7	7	2	4
8	8	1	3
9	9	0	0
10	7	7	7

STEP 2 : connect the points together with a (curved) line.

We have successfully represented the system's changes (of state) as a continuous motion in a space; a motion having a certain "course", "path", or "track" -- a certain trajectory. In fact, we can go so far as to make a single picture of our system that will re-present to us more about it than we could ever "see" by looking with eyes alone at our system at any one moment. One picture, in the form of a state-space trajectory, can tell its whole story -- show its entire history. We can see it "all at once" in a sense even stronger than that of seeing it, say, from all sides at once (spherical vision), at a single time, in space. We can depict it, using state-space practices, not only "from every angle" in the sense of "from all space", but in the sense of "from all time"* as well. We can put down in our state-space picture the track of all the states that our system "goes through" -- all the states that it instantaneously IS --during its entire life-time, its entire existence from "birth" to "death". But, unlike in direct perception, we can now "see" the object (system) in the unity of all its moments. We can see represented not only its three physical dimensions and motions in space (iff we have included the three spatial variables as among the essential state-variables), but also its 4th dimension in time. We display the system as a four dimensional object. For the first time, we grasp the system as a totality. We have formed and represented quantitatively its "Concept" ("Begriff") in Hegel's sense.** This totality, the entire state-space trajectory, the ensemble of its states --"the totality of its appearances"--is what Hegel means by the essence of a thing: all its faces, that it presents at different times; all its moments; its whole life--- "The truth is the whole." This is the truth of the system, this its reality, this its actual identity. Nothing less.

state-space "ballistics"

discuss how dimension >3 problem of various solution-attempts

= '4-d' seeing

This state-space trajectory (or the formula corresponding thereto ("generating" it, describing it, that it satisfies (all))) is also known as a mathematical model of the system-object. As we shall see, there are other ways of presenting this same model than as a trajectory-picture; it can also be formulated -- presented as a formula, an equation, called a state-function (state-vector-valued-function).

Now I can only "model" a system in the way so far described here if my life-time is substantially longer than that of the system I am "tracking". But what of systems whose life-times are significantly greater than my own -- take for example,

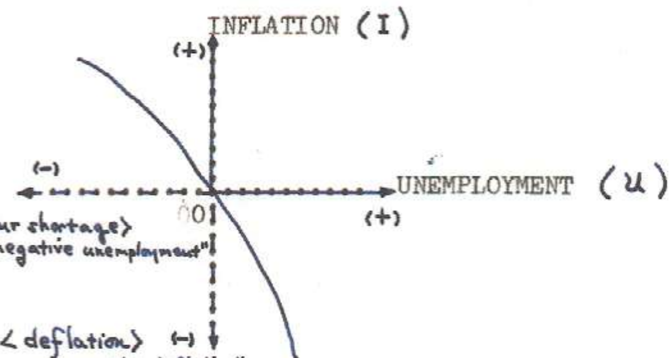
*("The system under consideration would be not only a spatial but also a temporal whole." -- Ludwig von Bertalanffy, General System Theory, p. 57 in Chapter 3: "Some System Concepts in Elementary Mathematical Consideration").

**(a "Concept" in this sense is itself a kind of mental model, a mental reproduction, of an "external" system-object).

human society. Is this technique useless here? Not any more than our minds are impotent before "larger" (in the 4th-dimensional sense) systems in general. This is where "Reason" comes in, takes over from and supersedes simple sensation; the rewrite senses operating alone, which would not "last" long enough to suffice here. With reason, ratiocination, and experimentation, we can ferret out the essence of the system, without having first to experience (interact with) all its states directly. We can write down this essence -- the "law" or invariant of the system -- as an equation of state (state-equation), reconstructing (that of) its possible past states which lie beyond our direct experience, and forecasting (that of) its probable future states, iff our grasp of its essence is right, and "tight". With regard to the system "capital", this was what Marx was trying to do with "Capital," volumes I, II, III, and IV (+): "it is the ultimate aim of this work, to lay bare the economic law of motion of modern society." (Capital I, NW p. 10,.).

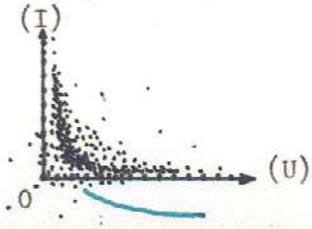
Now, suppose I am (a practical,) [if narrowly practical, proletarian] interested in the state and trajectory of the economy. Say The [only] two things that ["bother" interest me] most about the economy at first are (1) unemployment, and; (2) inflation. [So] Then these are the [only] two things I [bother] about, in terms of state-variables of the economy. These are the only two things I [care about] regarding the economy right now, and therefore these two measurements of the economy, as state-variables, are sufficient to characterize its "essence" in so far as I am concerned. I construct my state-space so:

(Figure 8.)



Suppose I began this project in a late-boom, pre-bust phase of the capitalist cycle, when the first signs of impending trouble were just beginning to show. Then, over a period of time, I sporadically consult the financial page of my newspaper, the government abstracts of economic statistics, etc. and start filling in points. After awhile, I end up with something like this:

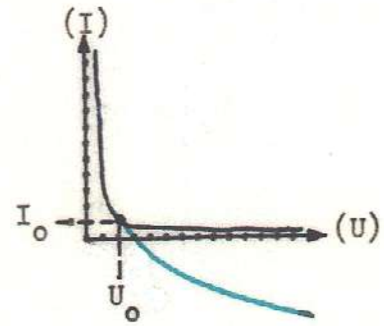
(Figure 9.)



(By the by; herein lies a good example -- timely, one might say -- of how "merely quantitative" representations in fact contain qualitative information: we know that when unemployment (U) crosses a certain quantitative threshold $U \geq U_x$ -- say unemployment $\geq 8\%$, a depression ensues).

Next, I draw a "trajectory" like this, as a sort of average-line between all the points in the cluster, duplicating its essential shape:

(Figure 10.)



This "trajectory" or curve -- I'm reluctant to call it a trajectory because we haven't established yet that the system-point actually moves along this curve as its path through this state-space -- depicts a certain relationship between unemployment and inflation. It says that, when unemployment is high, inflation is low, and that when inflation is high, unemployment is low. We might symbolize such a relationship in the following way:

$$U(t) \# I(t) \tag{1}$$

where the sign '#' sums up in one symbol the following transition of quantitative relations between U and I with increasing t:

$$\begin{aligned} U(t) &> I(t), \text{ for } t = t_o \\ U(t) &= I(t), \text{ for } t = t_o + \Delta t_o = t_1 \\ U(t) &< I(t), \text{ for } t = t_1 + \Delta t_1 = t_2 \end{aligned} \tag{2}$$

That is, the symbol '#' relating U and I designates the reversal of the inequality of U and I -- from "greater than" ($>$) to "less than" ($<$) -- mediated through equality; the negation of the negation of the original inequality. (For now, you can think of the sign '#' as a double-inequality sign -- the sign $\neq$ slashed again -- indicating that both of the two possible types of inequality, '$>$' and '$<$', are present in the quantitative relationship of U and I).

The curve itself is a locus of points that purports to define the total relationship between unemployment and inflation. This locus (set of connected points) tells you that, for any specific amount of unemployment, call it U_o, I_o, and none other, is the corresponding amount of inflation, and vice versa (see Figure 10. for an example of such a pair of points). U_o implies I_o. I_o implies U_o.

The graph itself is like a look-up table for translating one into the other: say for instance, 4 million unemployed "is also", "implies", "is another way of saying", or "is associated with" 10¢ per dollar of value per year of inflation on the price of the average commodity, etc. In general, every value of the state-variable U on its number-line or "line-space" ("state-line") is associated by the curve with a unique value of the state-variable I on its state-line, <u>and vice versa</u>. This is known in mathematics as a <u>relation</u>, which can be symbolized as R, so that the curve can be translated into a formula and written down like this:

$$I_o = R(U_o) \tag{3}$$

or, in general, as

$$I = R(U) \tag{4}$$

or as;

$$I(t) = R(\ U(t)\ ). \tag{5}$$

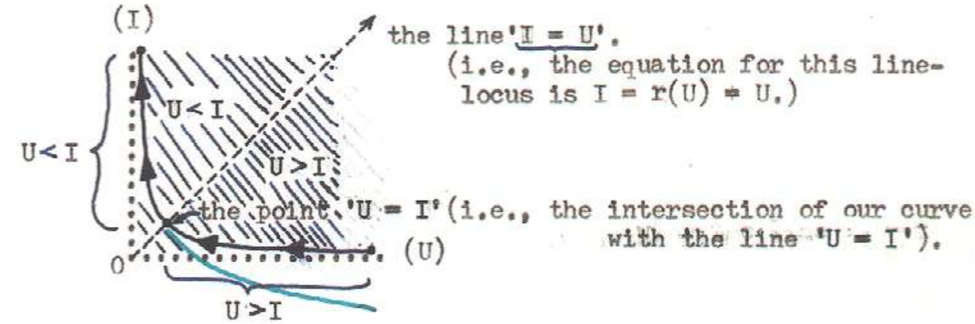
(Of course, this by itself only asserts that a relation exists between U and I, but does not by itself tell what that relation is. From the looks of the curve as it stands, this relation is probably something like $R(\ ) = (\ )^{-3}$ which in this case would mean $R(U) = U^{-3} = 1/U^3 = 1/U \cdot U \cdot U$. This means, for example, that if $U = 10$, $I = 1/10^3 = 1/1000$, and if $U = 1/10$, $I = 1/(1/10)^3 = 1/(1/1000) = 1000$. In general, $I = R(U) = U^n$, where n is an odd negative integer (-1, -3, -5, -7, etc.) would seem to be the general-form formula for this relationship. This digression is just to indicate that there is more to formulas than a mere general assertion of relationship, and that a formula can, on occasion, specify in one compact symbol like 'U^{-3}' the <u>whole</u> <u>locus</u> of a relation; all of its constituent points; the <u>whole</u> curve -- which gives an idea of the power and use-value of formulae).

The specific relation between unemployment and inflation, also known as the "Phillips curve", is an example of a special sub-type of mathematical <u>relation</u>, called a <u>function</u>. A function is a relation in which, given any value of one variable, one <u>and</u> <u>only</u> <u>one</u> value of the other variable is also determined*. In the other types of relations, in relations in general, a value of one variable may define one, two, <u>or</u> <u>more</u> "alternative" or "simultaneous" values of the other variable. Most of the state-formulae with which we will be concerned will be <u>relations</u> but <u>not</u> <u>functions</u> and, as we shall see, this is no accident.

Well, this is all fine and dandy, but we soon come up against some insufficiencies in our mathematical model of the economy so far. We have a kind of eternal, static relationship between I and U pictured here, an economic "law". Time, history, does not enter into the picture as a variable here. The law seems to exist as an "ideal relationshop" in "eternal space" (eternal state-space). As it stands, it doesn't tell us much historically; much about the temporal development of the system. We have, so far, simply put down points, without paying much attention to how they follow one another in time -- we haven't established their trajectory. Our model so far doesn't tell us much about the actual sequence of states; the order in which these system-points -- and therefore, the system-states -- succeed eachother in time. We need a way of organizing this state-space temporally, so that it gives us historical information at a glance. We need to be able to "see" the historical direction of the state-motion.

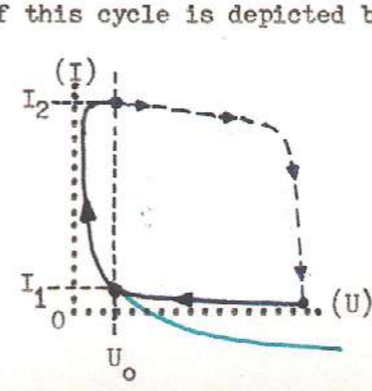
The formula '$U(t) \# I(t)$,' by the definition given in (2) above, already implies the following time direction of state-motion, because it says that $U > I \longrightarrow U = I \longrightarrow U < I$ in temporal succession:

(Figure 11.)

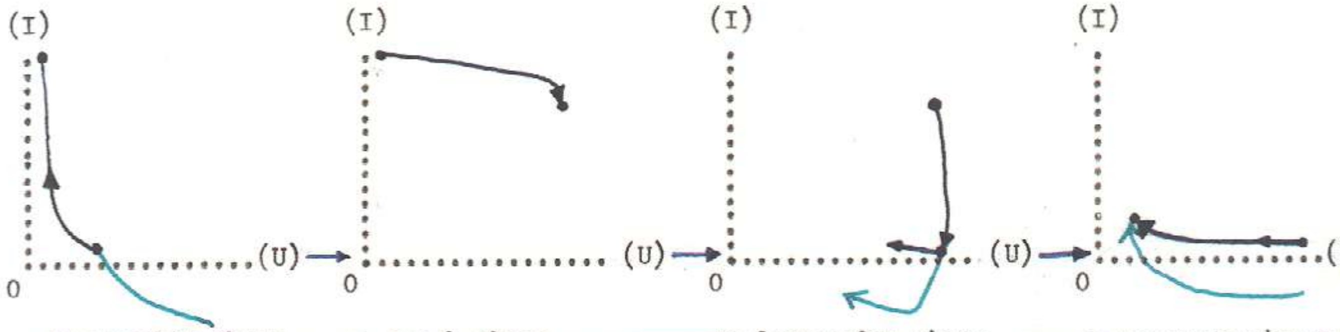


But does the movement stop at high-I, low-U? Suppose I notice, on recourse to the historical statistics, that we have in fact a cyclical motion, repeated periodically again and again over the last two centuries: rising inflation follows upon rising employment (=falling unemployment) as the economy enters a state of prosperity ("boom"); then soon inflation is rising a whole lot more while unemployment in falling only a little more; then follows a "crash" ("bust"), in which employment falls a whole lot faster than prices (inflation); but finally prices catch up in their fall and we have a state of high unemployment and low prices ("depression"), which is "in turn" followed by a recovery period in which unemployment falls much more rapidly than prices rise -- and thus we are back to our starting point, rapidly rising inflation plus near-full employment; and so on once again. One loop of this cycle is depicted below in Figures 12.

(Figure 12.)



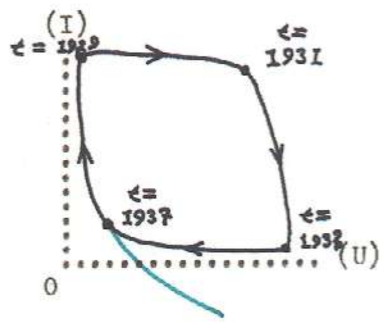
Figures 12. (cont'd.)



prosperity phase → crash phase → depression phase → recovery phase

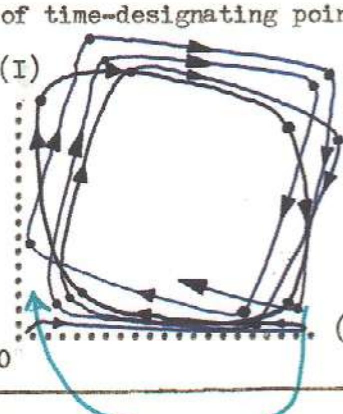
Note that our full curve for the cycle is no longer a function of U, but a relation, since to every value $U = U_0$ there now correspond two distinct values of I, I_1 and I_2. The most obvious, straight-forward way to build temporal, historical information "at-a-glance" into this model is thus to use arrows to indicate state-direction, and further more, to designate points on the trajectory by their t-values or "dates":

(Figure 13.)



But, suppose we now add in data gleaned from the historical statistics published in the U.S. government economic abstract from colonial times to the present,* for more or less the whole recorded history of U.S. capitalist development. Then our picture gets messy again, and the method of time-designating points proves impracticable:

(Figure 14.)



(attempt to time-designate points would lead to hopeless, illegible clutter).

*(Historical Statistics of the United States, Colonial Times to 1957 and Continuation).